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An almost sure central limit theorem for self-normalized products of sums of i.i.d. random variables ☆

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ABSTRACT

Let $X, X_1, X_2, \dots$ be a sequence of independent and identically distributed positive random variables with $EX = \mu > 0$. In this paper we show that the almost sure central limit theorem for self-normalized products of sums holds only under the assumptions that X belongs to the domain of attraction of the normal law.

1. Introduction and main results

Throughout this paper we assume $\{X, X_n;\ n \geqslant 1\}$ is a sequence of independent and identically distributed (i.i.d.) positive random variables with $EX = \mu > 0$ and define

$$S_n = \sum_{i=1}^n X_i, \qquad V_n^2 = \sum_{i=1}^n (X_i - \mu)^2 \quad \text{and} \quad \widetilde{V}_n^2 = \sum_{i=1}^n (X_i - \overline{X}_n)^2, \quad n = 1, 2, \dots,$$

where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. The limit theorems of products of $\prod_{j=1}^n S_j$ was initiated by Arnold and Villaseñor [1] who obtained the following version of the CLT for a sequence $\{X_n;\ n \geqslant 1\}$ of i.i.d. exponential r.v.'s with the mean equal to one

$$\left(\prod_{j=1}^n \frac{S_j}{j}\right)^{1/\sqrt{n}} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}}.$$

Here and in the sequel, $\mathcal{N}$ is a standard normal random variable. Their proof was heavily based on a very special property of exponential distributions. Later on, Rempala and Wesolowski [20] proved the following result.

Theorem A. *Let $\{X, X_n;\ n \geqslant 1\}$ be a sequence of i.i.d. positive random variables with $EX = \mu > 0$, and $\operatorname{Var} X = \sigma^2 < \infty$ and the coefficient of variation $\gamma = \sigma/\mu$. Then*

$$\left(\frac{\prod_{j=1}^n S_j}{n!\mu^n}\right)^{1/(\gamma\sqrt{n})} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}}. \tag{1.1}$$

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Recently, Qi [19] and Lu and Qi [16] extended (1.1) by assuming that the underlying distribution F is in the domain of attraction of a stable law with exponent $\alpha \in (1,2]$ and $\alpha = 1$, respectively. We next will recall the definition of the domain of a stable law.

A sequence of i.i.d. random variables $\{X, X_n;\ n \geqslant 1\}$ is said to be in the domain of attraction of a stable law $\mathcal{L}_\alpha$ if there exist constants $A_n > 0$ and $B_n \in R$ such that

$$\frac{S_n - B_n}{A_n} \xrightarrow{\mathrm{d}} \mathcal{L}_\alpha, \tag{1.2}$$

where $\mathcal{L}_\alpha$ is one of the stable distributions with index $\alpha \in (0,2]$. When $\alpha = 2$, $\{X, X_n;\ n \geqslant 1\}$ is said to be in the domain of attraction of the normal law.

In recent years, the limit theorems for self-normalized sums have received more and more attention. We refer to Griffin and Kuelbs [12] for laws of iterated logarithm, Bentkus and Götze [2] for Berry–Esseen inequalities, Lin [14] for Chung-type laws of iterated logarithm, Giné et al. [8] for the necessary and sufficient condition for the asymptotic normality, Shao [22–24] for large deviations, Csörgő et al. [6,7] for Darling–Erdös theorem and Donsker's theorem, Liu and Lin [15] for asymptotics for self-normalized random products of sums for mixing sequences. Pang et al. [17] obtained the following self-normalized products of sums for i.i.d. sequences.

Theorem B. *Let $\{X, X_n;\ n \geqslant 1\}$ be a sequence of i.i.d. positive random variables with $EX = \mu > 0$, and assume that X is in the domain of attraction of the normal law. Then*

$$\left(\frac{\prod_{j=1}^{n} S_j}{n!\mu^n}\right)^{\mu/\widetilde{V}_n} \xrightarrow{\mathrm{d}} \mathrm{e}^{\sqrt{2}\mathcal{N}}. \tag{1.3}$$

The almost sure central limit theorem (ASCLT) has been first introduced independently by Brosamler [4] and Schatte [21]. Since then many interesting results have been discovered in this field. The classical ASCLT states that when $EX = 0$, $\operatorname{Var}(X) = \sigma^2$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{\frac{S_k}{\sqrt{k}\sigma} \leqslant x\right\} = \Phi(x), \quad \text{a.s.} \tag{1.4}$$

for any $x \in R$. Here and in the sequel, $I\{\cdot\}$ denotes an indicator function and $\Phi(\cdot)$ is the distribution function of the standard normal random variable. Very recently, Gonchigdanzan and Rempala [11] proved the following ASCLT of products $\prod_{j=1}^{n} S_j$.

Theorem C. *Let $\{X, X_n;\ n \geqslant 1\}$ be a sequence of i.i.d. positive random variables with $EX = \mu > 0$, and $\operatorname{Var} X = \sigma^2 < \infty$ and the coefficient of variation $\gamma = \sigma/\mu$. Then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{\left(\frac{\prod_{j=1}^{k} S_j}{k!\mu^k}\right)^{1/(\gamma\sqrt{k})} \leqslant x\right\} = F_1(x), \quad a.s. \tag{1.5}$$

for any $x \in R$. Here and in the sequel, $F_1(\cdot)$ is the distribution function of the random variable $\mathrm{e}^{\sqrt{2}\mathcal{N}}$.

We also refer to Gonchigdanzan [9] for the ASCLT for the products of partial sums with stable distribution, Gonchigdanzan [10] for the almost sure functional limit theorem for the product of partial sums, Li and Wang [13] for the ASCLT for products of sums under association, Zhang et al. [25] for ASCLT for products of sums of partial sums under association.

The result in (1.3) shows that when $\sqrt{n}$ in the classical central limit theorem is replaced by an appropriate sequence of random variables then the central limit theorem holds under a weaker moment condition than in classical case. Thus, it is natural to ask whether a self-normalized version of the ASCLT analog to Theorem C could also be valid under the same weaker assumption. As the following theorem shows, the answer to this question is affirmative.

Theorem 1.1. *Let $\{X, X_n;\ n \geqslant 1\}$ be a sequence of i.i.d. positive random variables with $EX = \mu > 0$, and assume that X is in the domain of attraction of the normal law. Then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{\left(\frac{\prod_{j=1}^{k} S_j}{k!\mu^k}\right)^{\mu/V_n} \leqslant x\right\} = F_1(x), \quad a.s. \tag{1.6}$$

for any $x \in R$.

Theorem 1.2. *Let $\{X, X_n;\ n \geqslant 1\}$ be a sequence of i.i.d. positive random variables with $EX = \mu > 0$, and assume that X is in the domain of attraction of the normal law. Then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{\left(\frac{\prod_{j=1}^{k} S_j}{k!\mu^k}\right)^{\mu/\widetilde{V}_n} \leqslant x\right\} = F_1(x), \quad a.s. \tag{1.7}$$

for any $x \in R$.

Throughout the paper, C denotes a positive constant, which may take different values whenever it appears in different expressions. $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$.

2. Auxiliary lemmas

In this section, we introduce some important lemmas which are used to prove our theorems. At first, we introduce some notations that will be used throughout this paper. Let $l(x) = E(X-\mu)^2 I\{|X-\mu| \leqslant x\}$, $b = \inf\{x \geqslant 1:\ l(x) > 0\}$ and

$$\eta_j = \inf\left\{s:\ s \geqslant b+1,\ \frac{l(s)}{s^2} \leqslant \frac{1}{j}\right\}, \quad j = 1, 2, \ldots.$$

It is easy to see that $nl(\eta_n) \sim \eta_n^2$ as $n \to \infty$. In fact, by the definition of η_n, it is not difficult to obtain that $nl(\eta_n) \leqslant \eta_n^2$ for every $n \geqslant 1$. Next we want to show that $(n+1)l(\eta_n) \geqslant \eta_n^2$ for every $n \geqslant 1$. By the monotonicity of $l(x)$, the definition of η_n and $\eta_n \geqslant 2$, we have

$$nl(\eta_n) \geqslant nl\left(\eta_n - \frac{1}{n+1}\right) > \left(\eta_n - \frac{1}{n+1}\right)^2 \geqslant \eta_n^2 - \frac{1}{n+1}\eta_n^2 = \left(1 - \frac{1}{n+1}\right)\eta_n^2 = \frac{n}{n+1}\eta_n^2.$$

Thus we have

$$nl(\eta_n) \leqslant \eta_n^2 \leqslant (n+1)l(\eta_n) \quad \text{for every } n \geqslant 1.$$

Thus $nl(\eta_n) \sim \eta_n^2$ as $n \to \infty$. For every $1 \leqslant i \leqslant k \leqslant n$, let

$$\overline{X}_{ki} = (X_i - \mu) I\{|X_i - \mu| \leqslant \eta_k\}, \qquad \widetilde{X}_{ki} = (X_i - \mu) I\{|X_i - \mu| > \eta_k\},$$

$$X_{ki}^* = \overline{X}_{ki} - E\overline{X}_{ki}, \qquad \widetilde{X}_{ki}^* = \widetilde{X}_{ki} - E\widetilde{X}_{ki}, \qquad S_k^* = \sum_{i=1}^{k} X_{ki}^*, \qquad b_{i,k} = \sum_{l=i}^{k} \frac{1}{l},$$

$$Y_k = \sum_{i=1}^{k} b_{i,k} X_{ki}^*, \qquad \overline{Y}_k = \sum_{i=1}^{k} b_{i,k} \widetilde{X}_{ki}^*, \qquad \overline{V}_k^2 = \sum_{i=1}^{k} \overline{X}_{ki}^2.$$

The first of the following lemmas can be found in Csörgő et al. [7].

Lemma 2.1. *If $EX = 0$, then the following statements are equivalent*:

(a) $l(x) = EX^2 I\{|X| \leqslant x\}$ *is a slowly varying function at* ∞;
(b) $x^2 P(|X| > x) = o(l(x))$;
(c) $xE|X| I\{|X| > x\} = o(l(x))$;
(d) $E|X|^\alpha I\{|X| \leqslant x\} = o(x^{\alpha-2} l(x))$ *for* $\alpha > 2$.

Remark 2.1. The second condition (b) is well known to be equivalent to saying that X belongs to the domain of attraction of the normal law.

Lemma 2.2. *Let f be a real-valued function with $\sup_x |f(x)| \leqslant C$ and $\sup_x |f'(x)| \leqslant C$, then under the assumptions of Theorem 1.1, we have*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}\left[f\left(\frac{\overline{Y}_k}{\sqrt{kl(\eta_k)}}\right) - Ef\left(\frac{\overline{Y}_k}{\sqrt{kl(\eta_k)}}\right)\right] = 0, \quad a.s., \tag{2.1}$$

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{k} \frac{1}{k}\left[f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right) - Ef\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right)\right] = 0, \quad a.s., \tag{2.2}$$

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}\left[f\left(\frac{S_k^*}{k\sqrt{l(\eta_k)}}\right) - Ef\left(\frac{S_k^*}{k\sqrt{l(\eta_k)}}\right)\right] = 0, \quad a.s. \tag{2.3}$$

Proof. Let

$$T_n = \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\left[f\left(\frac{\overline{Y}_k}{\sqrt{kl(\eta_k)}}\right) - Ef\left(\frac{\overline{Y}_k}{\sqrt{kl(\eta_k)}}\right)\right] =: \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}Z_k. \tag{2.4}$$

It is obvious that

$$ET_n^2 = \frac{1}{\log^2 n}\left[\sum_{k=1}^{n}\frac{1}{k^2}EZ_k^2 + 2\sum_{k<j}\frac{1}{k}\frac{1}{j}EZ_kZ_j\right] =: \frac{1}{\log^2 n}[I_1 + I_2]. \tag{2.5}$$

By the fact that f is bounded, we have

$$\frac{1}{\log^2 n}|I_1| \leqslant \frac{C}{\log^2 n}\sum_{k=1}^{n}\frac{1}{k^2} \leqslant \frac{C}{\log n}. \tag{2.6}$$

For any $\alpha > 0$ and $j > k$, it is easy to check that

$$\log\frac{j}{k} \leqslant C\left(\frac{j}{k}\right)^{\alpha}. \tag{2.7}$$

Then under the assumptions of Lemma 2.2, for $1 \leqslant k < j \leqslant n$, we have

$$\begin{aligned}
|EY_kY_j| &= \left|\mathrm{Cov}\left(f\left(\frac{\overline{Y}_k}{\sqrt{kl(\eta_k)}}\right), f\left(\frac{\overline{Y}_j}{\sqrt{jl(\eta_j)}}\right)\right)\right| \\
&= \left|\mathrm{Cov}\left(f\left(\frac{\overline{Y}_k}{\sqrt{kl(\eta_k)}}\right), f\left(\frac{\overline{Y}_j}{\sqrt{jl(\eta_j)}}\right) - f\left(\frac{\overline{Y}_j - \sum_{i=1}^{k} b_{i,j}\widetilde{X}_{ji}^{*}}{\sqrt{jl(\eta_j)}}\right)\right)\right| \\
&\leqslant CE\left|f\left(\frac{\overline{Y}_j}{\sqrt{jl(\eta_j)}}\right) - f\left(\frac{\overline{Y}_j - \sum_{i=1}^{k} b_{i,j}\widetilde{X}_{ji}^{*}}{\sqrt{jl(\eta_j)}}\right)\right| \\
&\leqslant C\frac{E|\sum_{i=1}^{k} b_{i,j}\widetilde{X}_{ji}^{*}|}{\sqrt{jl(\eta_j)}} \leqslant C\frac{E|X-\mu|I\{|X-\mu| > \eta_j\}}{\sqrt{jl(\eta_j)}}\sum_{i=1}^{k} b_{i,j} \\
&\leqslant C\frac{E|X-\mu|I\{|X-\mu| > \eta_j\}}{\sqrt{jl(\eta_j)}}\left(\sum_{i=1}^{k}\sum_{l=i}^{k}\frac{1}{l} + kb_{k+1,j}\right) \\
&\leqslant C\frac{E|X-\mu|I\{|X-\mu| > \eta_j\}}{\sqrt{jl(\eta_j)}}\left(\sum_{l=1}^{k}\sum_{i=1}^{l}\frac{1}{l} + k\sum_{l=k+1}^{j}\frac{1}{l}\right) \\
&\leqslant C\frac{E|X-\mu|I\{|X-\mu| > \eta_j\}}{\sqrt{jl(\eta_j)}}\left(\sum_{l=1}^{k} l\frac{1}{l} + k\int_{k}^{j}\frac{1}{x}\,dx\right) \\
&\leqslant C\frac{E|X-\mu|I\{|X-\mu| > \eta_j\}}{\sqrt{jl(\eta_j)}}\left(k + k\log\frac{j}{k}\right) \\
&\leqslant C\frac{E|X-\mu|I\{|X-\mu| > \eta_j\}}{\sqrt{jl(\eta_j)}}k\left(\frac{j}{k}\right)^{\alpha},
\end{aligned} \tag{2.8}$$

where for our purpose, we fix $\alpha \in (0,1)$.

By Lemma 2.1 and $jl(\eta_j) \sim \eta_j^2$, there exists j_0 such that

$$E|X-\mu|I\{|X-\mu| > \eta_j\} \leqslant \frac{l(\eta_j)}{\eta_j}, \quad jl(\eta_j) \leqslant 2\eta_j^2, \tag{2.9}$$

for every $j > j_0$. Then by (2.8) and (2.9), we have

$$\begin{aligned}
\frac{1}{\log^2 n}|I_2| &\leqslant \frac{C}{\log^2 n}\sum_{k<j}\frac{1}{k}\frac{1}{j}|EY_kY_j| \leqslant \frac{C}{\log^2 n}\sum_{k<j}\frac{1}{k}\frac{1}{j}\frac{E|X-\mu|I\{|X-\mu|>\eta_j\}}{\sqrt{jl(\eta_j)}}k\left(\frac{j}{k}\right)^{\alpha} \\
&\leqslant \frac{C}{\log^2 n}\sum_{j=2}^{n}\sum_{k=1}^{j-1}\frac{1}{k^{\alpha}}\frac{1}{j^{3/2-\alpha}}\frac{E|X-\mu|I\{|X-\mu|>\eta_j\}}{\sqrt{l(\eta_j)}} \\
&\leqslant \frac{C}{\log^2 n}\sum_{j=2}^{j_0}\sum_{k=1}^{j-1}\frac{1}{k^{\alpha}}\frac{1}{j^{3/2-\alpha}}\frac{\mu}{\sqrt{l(\eta_j)}} + \frac{C}{\log^2 n}\sum_{j=j_0+1}^{n}\sum_{k=1}^{j-1}\frac{1}{k^{\alpha}}\frac{1}{j^{3/2-\alpha}}\frac{1}{\sqrt{l(\eta_j)}}\frac{l(\eta_j)}{\eta_j} \\
&\leqslant \frac{C}{\log^2 n} + \frac{C}{\log^2 n}\sum_{j=j_0+1}^{n}\sum_{k=1}^{j-1}\frac{1}{k^{\alpha}}\frac{1}{j^{3/2-\alpha}}\frac{\sqrt{l(\eta_j)}}{\eta_j} \\
&\leqslant \frac{C}{\log^2 n} + \frac{C}{\log^2 n}\sum_{j=j_0+1}^{n}\sum_{k=1}^{j-1}\frac{1}{k^{\alpha}}\frac{1}{j^{3/2-\alpha}}\frac{1}{\sqrt{j}} \\
&\leqslant \frac{C}{\log^2 n} + \frac{C}{\log^2 n}\sum_{j=j_0+1}^{n}\sum_{k=1}^{j-1}\frac{1}{k^{\alpha}}\frac{1}{j^{2-\alpha}} \\
&\leqslant \frac{C}{\log^2 n} + \frac{C}{\log^2 n}\sum_{j=j_0+1}^{n}\frac{1}{j} \leqslant \frac{C}{\log n}.
\end{aligned} \tag{2.10}$$

By (2.5), (2.6) and (2.10), we have

$$ET_n^2 \leqslant \frac{C}{\log n}.$$

Let $n_k = \mathrm{e}^{k^{\tau}}$, where $\tau > 1$. We get

$$\sum_{k=1}^{\infty} ET_{n_k}^2 < \infty.$$

By Borel–Cantelli lemma, we have

$$T_{n_k} \to 0, \quad \text{a.s. as } k \to \infty.$$

Note that

$$\frac{\log n_{k+1}}{\log n_k} = \frac{(k+1)^{\tau}}{k^{\tau}} \to 1, \quad \text{as } k \to \infty.$$

Since f is bounded, then for $n_k < n \leqslant n_{k+1}$, we obtain

$$\begin{aligned}
|T_n| &\leqslant \frac{1}{\log n_k}\left|\sum_{i=1}^{n_k}\frac{1}{i}\left[f\left(\frac{\overline{Y}_k}{\sqrt{kl(\eta_k)}}\right) - Ef\left(\frac{\overline{Y}_k}{\sqrt{kl(\eta_k)}}\right)\right]\right| + \frac{1}{\log n_k}\sum_{i=n_k}^{n_{k+1}}\frac{1}{i}\left|f\left(\frac{\overline{Y}_k}{\sqrt{kl(\eta_k)}}\right) - Ef\left(\frac{\overline{Y}_k}{\sqrt{kl(\eta_k)}}\right)\right| \\
&\leqslant |T_{n_k}| + C\left(\frac{\log n_{k+1}}{\log n_k} - 1\right) \to 0, \quad \text{a.s. as } n \to \infty.
\end{aligned}$$

Thus (2.1) is proved.

To prove (2.2), let

$$B_n = \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\left[f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right) - Ef\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right)\right]. \tag{2.11}$$

Then under the assumptions of Lemma 2.2, we see that

$$\begin{aligned}
EB_n^2 &\leqslant \frac{C}{\log^2 n}\sum_{k=1}^{n}\frac{1}{k^2}+\frac{2}{\log^2 n}\sum_{k<j}\frac{1}{k}\frac{1}{j}\left|\operatorname{Cov}\left(f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right), f\left(\frac{\overline{V}_j^2}{jl(\eta_j)}\right)\right)\right| \\
&\leqslant \frac{C}{\log n}+\frac{2}{\log^2 n}\sum_{k<j}\frac{1}{k}\frac{1}{j}\left|\operatorname{Cov}\left(f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right), f\left(\frac{\overline{V}_j^2}{jl(\eta_j)}\right)-f\left(\frac{\overline{V}_j^2-\sum_{i=1}^{k}\overline{X}_{ji}^2}{jl(\eta_j)}\right)\right)\right| \\
&\leqslant \frac{C}{\log n}+\frac{C}{\log^2 n}\sum_{k<j}\frac{1}{k}\frac{1}{j}\frac{E|\sum_{i=1}^{k}\overline{X}_{ji}^2|}{jl(\eta_j)} \leqslant \frac{C}{\log n}+\frac{C}{\log^2 n}\sum_{j=2}^{n}\sum_{k=1}^{j-1}\frac{1}{k}\frac{1}{j}\frac{kl(\eta_j)}{jl(\eta_j)} \\
&\leqslant \frac{C}{\log n}+\frac{C}{\log^2 n}\sum_{j=2}^{n}\frac{1}{j}\leqslant\frac{C}{\log n}.
\end{aligned} \tag{2.12}$$

The remaining part of the proof is similar to that of (2.1), then we get (2.2).

To prove (2.3), let

$$F_n=\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\left[f\left(\frac{S_k^*}{k\sqrt{l(\eta_k)}}\right)-Ef\left(\frac{S_k^*}{k\sqrt{l(\eta_k)}}\right)\right]. \tag{2.13}$$

Then under the assumptions of Lemma 2.2, we obtain

$$\begin{aligned}
EF_n^2 &\leqslant \frac{C}{\log^2 n}\sum_{k=1}^{n}\frac{1}{k^2}+\frac{2}{\log^2 n}\sum_{k<j}\frac{1}{k}\frac{1}{j}\left|\operatorname{Cov}\left(f\left(\frac{S_k^*}{k\sqrt{l(\eta_k)}}\right), f\left(\frac{S_j^*}{j\sqrt{l(\eta_j)}}\right)\right)\right| \\
&\leqslant \frac{C}{\log n}+\frac{C}{\log^2 n}\sum_{k<j}\frac{1}{k}\frac{1}{j}\left|\operatorname{Cov}\left(f\left(\frac{S_k^*}{k\sqrt{l(\eta_k)}}\right), f\left(\frac{S_j^*}{j\sqrt{l(\eta_j)}}\right)-f\left(\frac{S_j^*-\sum_{i=1}^{k}X_{ji}^*}{j\sqrt{l(\eta_j)}}\right)\right)\right| \\
&\leqslant \frac{C}{\log n}+\frac{C}{\log^2 n}\sum_{k<j}\frac{1}{k}\frac{1}{j}\frac{E|\sum_{i=1}^{k}X_{ji}^*|}{j\sqrt{l(\eta_j)}} \\
&\leqslant \frac{C}{\log n}+\frac{C}{\log^2 n}\sum_{k<j}\frac{1}{k}\frac{1}{j}\frac{(E|\sum_{i=1}^{k}X_{ji}^*|^2)^{1/2}}{j\sqrt{l(\eta_j)}} \\
&\leqslant \frac{C}{\log n}+\frac{C}{\log^2 n}\sum_{j=2}^{n}\sum_{k=1}^{j-1}\frac{1}{k}\frac{1}{j^2}\frac{(kl(\eta_j))^{1/2}}{\sqrt{l(\eta_j)}} \\
&\leqslant \frac{C}{\log n}+\frac{C}{\log^2 n}\sum_{j=2}^{n}\sum_{k=1}^{j-1}\frac{1}{k^{1/2}}\frac{1}{j^2} \\
&\leqslant \frac{C}{\log n}+\frac{C}{\log^2 n}\sum_{j=2}^{n}\frac{1}{j^{3/2}}\leqslant\frac{C}{\log n}.
\end{aligned} \tag{2.14}$$

Then by the same argument as in (2.1), we get (2.3). □

Lemma 2.3. *Under the assumptions of Theorem* 1.1, *we have*

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\left[I\left\{\bigcup_{j=1}^{k}\left(|X_j-\mu|>\eta_k\right)\right\}-EI\left\{\bigcup_{j=1}^{k}\left(|X_j-\mu|>\eta_k\right)\right\}\right]=0,\quad a.s. \tag{2.15}$$

Proof. Let

$$D_n=\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\left[I\left\{\bigcup_{j=1}^{k}\left(|X_j-\mu|>\eta_k\right)\right\}-EI\left\{\bigcup_{j=1}^{k}\left(|X_j-\mu|>\eta_k\right)\right\}\right]=:\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\zeta_k.$$

It is easy to prove that

$$ED_n^2 = \frac{1}{\log^2 n}\left[\sum_{k=1}^{n} \frac{1}{k^2} E\zeta_k^2 + 2\sum_{i<k} \frac{1}{i}\frac{1}{k} E\zeta_i\zeta_k\right] =: \frac{1}{\log^2 n}[J_1 + J_2]. \tag{2.16}$$

By the fact that $\{\zeta_i\}$ is bounded, we have

$$\frac{1}{\log^2 n}|J_1| \leqslant \frac{C}{\log^2 n}\sum_{i=1}^{n}\frac{1}{i^2} \leqslant \frac{C}{\log n}. \tag{2.17}$$

It is known that $|I\{A \cup B\} - I\{B\}| \leqslant I\{A\}$ for any sets A and B, then we have

$$\begin{aligned}
\frac{1}{\log^2 n}|J_2| &\leqslant \frac{C}{\log^2 n}\sum_{i<k}\frac{1}{i}\frac{1}{k}|E\zeta_i\zeta_k| \\
&\leqslant \frac{C}{\log^2 n}\sum_{i<k}\frac{1}{i}\frac{1}{k}\left|\mathrm{Cov}\left(I\left\{\bigcup_{j=1}^{i}\big(|X_j-\mu|>\eta_i\big)\right\}, I\left\{\bigcup_{j=1}^{k}\big(|X_j-\mu|>\eta_k\big)\right\}\right)\right| \\
&\leqslant \frac{C}{\log^2 n}\sum_{i<k}\frac{1}{i}\frac{1}{k}\left|\mathrm{Cov}\left(I\left\{\bigcup_{j=1}^{i}\big(|X_j-\mu|>\eta_i\big)\right\}, I\left\{\bigcup_{j=1}^{k}\big(|X_j-\mu|>\eta_k\big)\right\} - I\left\{\bigcup_{j=i+1}^{k}\big(|X_j-\mu|>\eta_k\big)\right\}\right)\right| \\
&\leqslant \frac{C}{\log^2 n}\sum_{i<k}\frac{1}{i}\frac{1}{k}E\left|I\left\{\bigcup_{j=1}^{k}\big(|X_j-\mu|>\eta_k\big)\right\} - I\left\{\bigcup_{j=i+1}^{k}\big(|X_j-\mu|>\eta_k\big)\right\}\right| \\
&\leqslant \frac{C}{\log^2 n}\sum_{i<k}\frac{1}{i}\frac{1}{k}E\left|I\left\{\bigcup_{j=1}^{i}\big(|X_j-\mu|>\eta_k\big)\right\}\right| \\
&\leqslant \frac{C}{\log^2 n}\sum_{i<k}\frac{1}{i}\frac{1}{k} iP\big(|X-\mu|>\eta_k\big) \\
&\leqslant \frac{C}{\log^2 n}\sum_{k=2}^{n}\sum_{i=1}^{k-1}\frac{1}{k}P\big(|X-\mu|>\eta_k\big) \\
&\leqslant \frac{C}{\log^2 n}\sum_{k=2}^{n}P\big(|X-\mu|>\eta_k\big).
\end{aligned} \tag{2.18}$$

By Lemma 2.1 and $kl(\eta_k) \sim \eta_k^2$, there exists k_0 such that

$$P\big(|X-\mu|>\eta_k\big) \leqslant \frac{l(\eta_k)}{\eta_k^2}, \quad kl(\eta_k) \leqslant 2\eta_k^2, \tag{2.19}$$

for every $k > k_0$. Then by (2.18) and (2.19), we have

$$\begin{aligned}
\frac{1}{\log^2 n}|J_2| &\leqslant \frac{C}{\log^2 n}\sum_{k=2}^{k_0}P\big(|X-\mu|>\eta_k\big) + \frac{C}{\log^2 n}\sum_{k=k_0+1}^{n}P\big(|X-\mu|>\eta_k\big) \\
&\leqslant \frac{C}{\log^2 n} + \frac{C}{\log^2 n}\sum_{k=k_0+1}^{n}\frac{l(\eta_k)}{\eta_k^2} \leqslant \frac{C}{\log^2 n} + \frac{C}{\log^2 n}\sum_{k=k_0+1}^{n}\frac{1}{k} \leqslant \frac{C}{\log n}.
\end{aligned} \tag{2.20}$$

Combining (2.16), (2.17) with (2.20), we have

$$ED_n^2 \leqslant \frac{C}{\log n}.$$

The remaining part of the proof is similar to that of (2.1). So the proof is complete. □

Lemma 2.4. *Under the assumptions of Theorem* 1.1, *we have*

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{Y_k}{\sqrt{2kl(\eta_k)}} \leqslant x\right\} = \Phi(x), \quad a.s. \tag{2.21}$$

for any $x \in R$.

Proof. By the formula (2.10) in Pang et al. [17], we have

$$\frac{Y_k}{\sqrt{2kl(\eta_k)}} \xrightarrow{\mathrm{d}} \mathcal{N}. \tag{2.22}$$

Let f be a bounded Lipschitz function and have a Radon–Nikodyn derivative h bounded by C. From (2.22), we have

$$Ef\left(\frac{Y_k}{\sqrt{2kl(\eta_k)}}\right) \to Ef\big(\mathcal{N}(0,1)\big), \quad \text{as } k \to \infty. \tag{2.23}$$

On the other hand, note that (2.21) is equivalent to

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} f\left(\frac{Y_k}{\sqrt{2kl(\eta_k)}}\right) = Ef\big(\mathcal{N}(0,1)\big), \quad \text{a.s.} \tag{2.24}$$

from Section 2 of Peligrad and Shao [18] and Theorem 7.1 of Billingsley [3]. Hence, to prove (2.21), it suffices to show that

$$R_n =: \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left[f\left(\frac{Y_k}{\sqrt{2kl(\eta_k)}}\right) - Ef\left(\frac{Y_k}{\sqrt{2kl(\eta_k)}}\right)\right] \to 0, \quad \text{a.s. as } n \to \infty. \tag{2.25}$$

By the formula (4) in Rempala and Wesolowski [20], we know

$$\sum_{i=1}^{k} b_{i,k}^2 = 2k - b_{1,k} \leqslant 2k. \tag{2.26}$$

By (2.7) and (2.26), similarly to (2.8), for $k < j$, we get

$$\begin{aligned}
\left|\mathrm{Cov}\left(f\left(\frac{Y_k}{\sqrt{2kl(\eta_k)}}\right), f\left(\frac{Y_j}{\sqrt{2jl(\eta_j)}}\right)\right)\right| &= \left|\mathrm{Cov}\left(f\left(\frac{Y_k}{\sqrt{2kl(\eta_k)}}\right), f\left(\frac{Y_j}{\sqrt{2jl(\eta_j)}}\right) - f\left(\frac{Y_j - \sum_{i=1}^{k} b_{i,j} X_{ji}^*}{\sqrt{2jl(\eta_j)}}\right)\right)\right| \\
&\leqslant C \frac{E|\sum_{i=1}^{k} b_{i,j} X_{ji}^*|}{\sqrt{2jl(\eta_j)}} \leqslant C \frac{1}{\sqrt{2jl(\eta_j)}} \left(E\left|\sum_{i=1}^{k} b_{i,j} X_{ji}^*\right|^2\right)^{1/2} \\
&\leqslant \frac{C}{\sqrt{2jl(\eta_j)}} \left(\sum_{i=1}^{k} b_{i,j}^2\right)^{1/2} l^{1/2}(\eta_j) \leqslant \frac{C}{\sqrt{j}} \left(\sum_{i=1}^{k} (b_{i,k} + b_{k+1,j})^2\right)^{1/2} \\
&\leqslant \frac{C}{\sqrt{j}} \left(\sum_{i=1}^{k} b_{i,k}^2 + \sum_{i=1}^{k} b_{k+1,j}^2\right)^{1/2} \leqslant \frac{C}{\sqrt{j}} \left(k + k\log^2\left(\frac{j}{k}\right)\right)^{1/2} \\
&\leqslant \frac{C}{\sqrt{j}} k^{1/2} \log\frac{j}{k} \leqslant C\sqrt{\frac{k}{j}} \left(\frac{j}{k}\right)^{\alpha},
\end{aligned} \tag{2.27}$$

where we can select $\alpha \in (0, 1/2)$. Then it is easy to prove

$$ER_n^2 \leqslant \frac{C}{\log n}.$$

The remaining part of the proof is similar to that of (2.1). So the proof is complete. □

3. Proof of the main theorems

Proof of Theorem 1.1. Let $C_i = S_i/(i\mu)$, we have

$$\frac{\mu}{\sqrt{2kl(\eta_k)}} \sum_{i=1}^{k} (C_i - 1) = \frac{1}{\sqrt{2kl(\eta_k)}} \left[\sum_{j=1}^{k} \sum_{l=j}^{k} \frac{1}{l} X_{kj}^* + \sum_{j=1}^{k} \sum_{l=j}^{k} \frac{1}{l} \widetilde{X}_{kj}^*\right] = \frac{1}{\sqrt{2kl(\eta_k)}} [Y_k + \overline{Y}_k]. \tag{3.1}$$

Note that in order to prove (1.6), it is sufficient to show that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{\frac{\mu}{\sqrt{2}V_k} \sum_{i=1}^{k} \log C_i \leqslant x\right\} = \Phi(x), \quad \text{a.s.} \tag{3.2}$$

for any $x \in R$. Furthermore, (3.2) is equivalent to the following two inequalities

$$\limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{ \frac{\mu}{\sqrt{2}V_k} \sum_{i=1}^{k} \log C_i \leqslant x \right\} \leqslant \Phi(x), \quad \text{a.s.} \tag{3.3}$$

and

$$\liminf_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{ \frac{\mu}{\sqrt{2}V_k} \sum_{i=1}^{k} \log C_i \leqslant x \right\} \geqslant \Phi(x), \quad \text{a.s.} \tag{3.4}$$

for any $x \in R$.

Firstly, we prove (3.3). For $x \geqslant 0$ and $0 < \delta_1 < 1/2$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{ \frac{\mu}{\sqrt{2}V_k} \sum_{i=1}^{k} \log C_i \leqslant x \right\} \\
&\leqslant \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left[I\left\{ \mu \sum_{i=1}^{k} \log C_i \leqslant x\sqrt{2(1+\delta_1)kl(\eta_k)} \right\} + I\left\{ V_k^2 > (1+\delta_1)kl(\eta_k) \right\} \right] \\
&\leqslant \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left[I\left\{ \mu \sum_{i=1}^{k} \log C_i \leqslant x\sqrt{2(1+\delta_1)kl(\eta_k)} \right\} + I\left\{ \overline{V}_k^2 > (1+\delta_1)kl(\eta_k) \right\} \right. \\
&\quad \left. + I\left\{ \bigcup_{j=1}^{k} \left\{ |X_j - \mu| > \eta_n \right\} \right\} \right] \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{3.5}$$

For $x < 0$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{ \frac{\mu}{\sqrt{2}V_k} \sum_{i=1}^{k} \log C_i \leqslant x \right\} \\
&\leqslant \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left[I\left\{ \mu \sum_{i=1}^{k} \log C_i \leqslant x\sqrt{2(1-\delta_1)kl(\eta_k)} \right\} + I\left\{ V_k^2 < (1-\delta_1)kl(\eta_k) \right\} \right] \\
&\leqslant \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left[I\left\{ \mu \sum_{i=1}^{k} \log C_i \leqslant x\sqrt{2(1+\delta_1)kl(\eta_k)} \right\} + I\left\{ \overline{V}_k^2 < (1-\delta_1)kl(\eta_k) \right\} \right. \\
&\quad \left. + I\left\{ \bigcup_{j=1}^{k} \left\{ |X_j - \mu| > \eta_n \right\} \right\} \right] \\
&=: J_1 + J_2 + J_3.
\end{aligned} \tag{3.6}$$

We need only to prove (3.3) holds for $x \geqslant 0$. For $x < 0$, we have the same conclusion. Let f_1 be real-valued function such that

$$I\left\{ |x| \geqslant 1+\delta_1 \right\} \leqslant f_1(x) \leqslant I\left\{ |x| \geqslant 1+\delta_1/2 \right\} \quad \text{and} \quad \sup_x \left| f_1'(x) \right| < \infty.$$

By Lemma 2.1 and $kl(\eta_k) \sim \eta_k^2$, for arbitrary $\epsilon > 0$, there exists k_1 such that

$$E|X|I\left\{ |X| \leqslant \eta_k \right\} \leqslant \epsilon l(\eta_k)/\eta_k, \quad kl(\eta_k) \geqslant \eta_k^2/2, \tag{3.7}$$

for every $k > k_1$. By Lemma 2.2 and (3.7), we have

$$\begin{aligned}
I_2 &\leqslant \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} f_1\left(\frac{\overline{V}_k^2}{kl(\eta_k)} \right) \\
&\leqslant \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left[f_1\left(\frac{\overline{V}_k^2}{kl(\eta_k)} \right) - Ef_1\left(\frac{\overline{V}_k^2}{kl(\eta_k)} \right) \right] + \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} Ef_1\left(\frac{\overline{V}_k^2}{kl(\eta_k)} \right)
\end{aligned}$$

$$\leqslant \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} E f_1\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right) \leqslant \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} P\left(\overline{V}_k^2 > (1+\delta/2)kl(\eta_k)\right)$$
$$\leqslant \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \frac{E(\overline{V}_k^2)^2}{(1+\delta/2)^2 k^2 l^2(\eta_k)} \leqslant \limsup_{n\to\infty} \frac{C}{\log n} \sum_{k=1}^{n} \frac{1}{k} \frac{kE|X-\mu|^4 I\{|X-\mu| \leqslant \eta_k\}}{(1+\delta/2)^2 k^2 l^2(\eta_k)}$$
$$\leqslant \limsup_{n\to\infty} \frac{C}{\log n} \sum_{k=1}^{k_1} \frac{E|X-\mu|^4 I\{|X-\mu| \leqslant \eta_k\}}{k^2 l^2(\eta_k)} + \limsup_{n\to\infty} \frac{C}{\log n} \sum_{k=k_1+1}^{n} \frac{E|X-\mu|^4 I\{|X-\mu| \leqslant \eta_k\}}{k^2 l^2(\eta_k)}$$
$$\leqslant \limsup_{n\to\infty} \frac{C}{\log n} + \limsup_{n\to\infty} \frac{C}{\log n} \sum_{k=k_1+1}^{n} \frac{\epsilon \eta_k^2 l(\eta_k)}{k^2 l^2(\eta_k)}$$
$$\leqslant \limsup_{n\to\infty} \frac{C}{\log n} \sum_{k=k_1+1}^{n} \frac{\epsilon}{k} = \epsilon, \quad \text{a.s.}$$

Now, letting $\epsilon \to 0$, we obtain

$$I_2 = 0, \quad \text{a.s.} \tag{3.8}$$

By Lemma 2.1 and Lemma 2.3, following the proof of (3.8), we have

$$I_3 = 0, \quad \text{a.s.} \tag{3.9}$$

Now we estimate I_1. Note that $E|X|^p < \infty$ for all $1 < p < 2$ (since X belongs to the domain of attraction of the normal law). For our purpose, we fix $4/3 < p < 2$. By Marcinkiewicz–Zygmund's strong law of large numbers (see Chow and Teicher [5, p. 125]), for i large enough, we have

$$|C_i - 1| \leqslant i^{1/p-1}, \quad \text{a.s.}$$

It is easy to see that $\log(1+x) - x = O(x^2)$ as $x \to 0$. Thus

$$\left|\sum_{i=1}^{k} \log C_i - \sum_{i=1}^{k} (C_i - 1)\right| \leqslant C \sum_{i=1}^{k} (C_i - 1)^2 \leqslant C k^{2/p-1}, \quad \text{a.s.}$$

Let $\{x_n\}, \{y_n\}$ be two sequences of real numbers, then the following inequalities are satisfied

$$\liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \leqslant \limsup_{n\to\infty} (x_n + y_n) \leqslant \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n, \tag{3.10}$$
$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \leqslant \liminf_{n\to\infty} (x_n + y_n) \leqslant \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n. \tag{3.11}$$

By Lemma 2.1 and (3.10), for arbitrary small $\varepsilon > 0$, there exists $k_0 = k_0(\omega, \varepsilon, x)$ such that for $k > k_0$,

$$\limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=k_0+1}^{n} \frac{1}{k} I\left\{\frac{\mu}{\sqrt{2(1+\delta_1)kl(\eta_k)}} \sum_{i=1}^{k} (C_i - 1) \leqslant x - \varepsilon\right\}$$
$$\leqslant \liminf_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{k_0} \frac{1}{k} I\left\{\frac{\mu}{\sqrt{2(1+\delta_1)kl(\eta_k)}} \sum_{i=1}^{k} \log C_i \leqslant x\right\}$$
$$+ \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=k_0+1}^{n} \frac{1}{k} I\left\{\frac{\mu}{\sqrt{2(1+\delta_1)kl(\eta_k)}} \sum_{i=1}^{k} \log C_i \leqslant x\right\}$$
$$\leqslant \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{\frac{\mu}{\sqrt{2(1+\delta_1)kl(\eta_k)}} \sum_{i=1}^{k} \log C_i \leqslant x\right\} = I_1$$
$$\leqslant \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{k_0} \frac{1}{k} I\left\{\frac{\mu}{\sqrt{2(1+\delta_1)kl(\eta_k)}} \sum_{i=1}^{k} \log C_i \leqslant x\right\}$$

$$+\limsup_{n\to\infty}\frac{1}{\log n}\sum_{k=k_0+1}^{n}\frac{1}{k}I\left\{\frac{\mu}{\sqrt{2(1+\delta_1)kl(\eta_k)}}\sum_{i=1}^{k}\log C_i\leqslant x\right\}$$
$$\leqslant\limsup_{n\to\infty}\frac{1}{\log n}\sum_{k=k_0+1}^{n}\frac{1}{k}I\left\{\frac{\mu}{\sqrt{2(1+\delta_1)kl(\eta_k)}}\sum_{i=1}^{k}(C_i-1)\leqslant x+\varepsilon\right\}. \tag{3.12}$$

For any $0<\delta_2<1/2$, we have

$$\limsup_{n\to\infty}\frac{1}{\log n}\sum_{k=k_0+1}^{n}\frac{1}{k}I\left\{\frac{\mu}{\sqrt{2(1+\delta_1)kl(\eta_k)}}\sum_{i=1}^{k}(C_i-1)\leqslant x+\varepsilon\right\}$$
$$\leqslant\limsup_{n\to\infty}\frac{1}{\log n}\sum_{k=k_0+1}^{n}\frac{1}{k}I\left\{\frac{Y_k}{\sqrt{2(1+\delta_1)kl(\eta_k)}}\leqslant x+\varepsilon+\delta_2\right\}+\limsup_{n\to\infty}\frac{1}{\log n}\sum_{k=k_0+1}^{n}\frac{1}{k}I\left\{\frac{|\overline{Y}_k|}{\sqrt{2(1+\delta_1)kl(\eta_k)}}\geqslant\delta_2\right\}. \tag{3.13}$$

By Lemma 2.4, we have

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=k_0+1}^{n}\frac{1}{k}I\left\{\frac{Y_k}{\sqrt{2(1+\delta_1)kl(\eta_k)}}\leqslant x+\varepsilon+\delta_2\right\}=\Phi\big(\sqrt{1+\delta_1}(x+\varepsilon+\delta_2)\big),\quad \text{a.s.} \tag{3.14}$$

By Lemma 2.1 and Lemma 2.2, following the proof of (3.8), we have

$$\limsup_{n\to\infty}\frac{1}{\log n}\sum_{k=k_0+1}^{n}\frac{1}{k}I\left\{\frac{|\overline{Y}_k|}{\sqrt{2(1+\delta_1)kl(\eta_k)}}\geqslant\delta_2\right\}=0,\quad \text{a.s.} \tag{3.15}$$

By (3.5)–(3.17) and letting $\delta_1\to 0,\delta_2\to 0$ and $\varepsilon\to 0$, then we obtain that (3.3) holds for $x\geqslant 0$. Thus (3.3) holds for all $x\in R$.

Next we want to prove (3.4). For $x\geqslant 0$ and $0<\delta_1<1/2$, we have

$$\liminf_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{\mu}{\sqrt{2}V_k}\sum_{i=1}^{k}\log C_i\leqslant x\right\}$$
$$\geqslant\liminf_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\left[I\left\{\mu\sum_{i=1}^{k}\log C_i\leqslant x\sqrt{2(1-\delta_1)kl(\eta_k)}\right\}-I\left\{V_k^2<(1-\delta_1)kl(\eta_k)\right\}\right]$$
$$\geqslant\liminf_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\left[I\left\{\mu\sum_{i=1}^{k}\log C_i\leqslant x\sqrt{2(1-\delta_1)kl(\eta_k)}\right\}-I\left\{\overline{V}_k^2<(1-\delta_1)kl(\eta_k)\right\}\right.$$
$$\left.-I\left\{\bigcup_{j=1}^{k}\left\{|X_j-\mu|>\eta_n\right\}\right\}\right].$$

For $x<0$, we have

$$\liminf_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{\mu}{\sqrt{2}V_k}\sum_{i=1}^{k}\log C_i\leqslant x\right\}$$
$$\geqslant\liminf_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\left[I\left\{\mu\sum_{i=1}^{k}\log C_i\leqslant x\sqrt{2(1+\delta_1)kl(\eta_k)}\right\}+I\left\{V_k^2>(1+\delta_1)kl(\eta_k)\right\}\right]$$
$$\geqslant\liminf_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\left[I\left\{\mu\sum_{i=1}^{k}\log C_i\leqslant x\sqrt{2(1+\delta_1)kl(\eta_k)}\right\}-I\left\{\overline{V}_k^2>(1+\delta_1)kl(\eta_k)\right\}\right.$$
$$\left.-I\left\{\bigcup_{j=1}^{k}\left\{|X_j-\mu|>\eta_n\right\}\right\}\right].$$

The remaining part of the proof is similar to the proof of (3.3), thus the proof is complete. □

Proof of Theorem 1.2. The proof of Theorem 1.2 is similar to that of Theorem 1.1, and we will give a short explanation.

In order to prove (1.7), it is sufficient to show that

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{\mu}{\sqrt{2}\widetilde{V}_k}\sum_{i=1}^{k}\log C_i\leqslant x\right\}=\Phi(x),\quad \text{a.s.}\tag{3.16}$$

for any $x\in R$. However, (3.16) is equivalent to the following two inequalities

$$\limsup_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{\mu}{\sqrt{2}\widetilde{V}_k}\sum_{i=1}^{k}\log C_i\leqslant x\right\}\leqslant\Phi(x),\quad \text{a.s.}\tag{3.17}$$

and

$$\liminf_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{\mu}{\sqrt{2}\widetilde{V}_k}\sum_{i=1}^{k}\log C_i\leqslant x\right\}\geqslant\Phi(x),\quad \text{a.s.}\tag{3.18}$$

for any $x\in R$.

Firstly, we want to prove (3.17). For $x\geqslant 0$, note that $\widetilde{V}_n^2=V_n^2-n(\overline{X}_n-\mu)^2\leqslant V_n^2$, then by (3.3), we have

$$\limsup_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{\mu}{\sqrt{2}\widetilde{V}_k}\sum_{i=1}^{k}\log C_i\leqslant x\right\}\leqslant\limsup_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{\mu}{\sqrt{2}V_k}\sum_{i=1}^{k}\log C_i\leqslant x\right\}\leqslant\Phi(x),\quad \text{a.s.}\tag{3.19}$$

For $x<0$, $0<\delta_3<1/2$ and $0<\delta_4<1/2$, note that $\widetilde{V}_n^2=V_n^2-n(\overline{X}_n-\mu)^2$, we have

$$\begin{aligned}
I\left\{\frac{\mu}{\sqrt{2}\widetilde{V}_k}\sum_{i=1}^{k}\log C_i\leqslant x\right\}&=I\left\{\frac{\mu}{\sqrt{2}\widetilde{V}_k}\sum_{i=1}^{k}\log C_i\leqslant x,\ \widetilde{V}_k^2\geqslant(1-\delta_3)V_k^2\right\}\\
&\quad+I\left\{\frac{\mu}{\sqrt{2}\widetilde{V}_k}\sum_{i=1}^{k}\log C_i\leqslant x,\ \widetilde{V}_k^2<(1-\delta_3)V_k^2\right\}\\
&\leqslant I\left\{\frac{\mu}{\sqrt{2}V_k}\sum_{i=1}^{k}\log C_i\leqslant(1-\delta_3)x\right\}+I\left\{\widetilde{V}_k^2<(1-\delta_3)V_k^2\right\}\\
&=I\left\{\frac{\mu}{\sqrt{2}V_k}\sum_{i=1}^{k}\log C_i\leqslant(1-\delta_3)x\right\}+I\left\{k(\overline{X}_k-\mu)^2>\delta_3V_k^2\right\}\\
&\leqslant I\left\{\frac{\mu}{\sqrt{2}V_k}\sum_{i=1}^{k}\log C_i\leqslant(1-\delta_3)x\right\}+I\left\{k(\overline{X}_k-\mu)^2\geqslant\delta_3(1-\delta_4)kl(\eta_k)\right\}\\
&\quad+I\left\{V_k^2<(1-\delta_4)kl(\eta_k)\right\}\\
&\leqslant I\left\{\frac{\mu}{\sqrt{2}V_k}\sum_{i=1}^{k}\log C_i\leqslant(1-\delta_3)x\right\}+I\left\{|S_k^*|\geqslant k\sqrt{\delta_3(1-\delta_4)l(\eta_k)}\right\}\\
&\quad+I\left\{\overline{V}_k^2<(1-\delta_4)kl(\eta_k)\right\}+2I\left\{\bigcup_{j=1}^{k}\left\{|X_j-\mu|>\eta_k\right\}\right\}\\
&=:T_{11}+T_{12}+T_{13}+2T_{14}.
\end{aligned}\tag{3.20}$$

The remaining part of the proof is similar to that of Theorem 1.1, so we omit it here. □

References

[1] B.C. Arnold, J.A. Villaseñor, The asymptotic distribution of sums of records, Extremes 1 (1998) 351–363.
[2] V. Bentkus, F. Götze, The Berry–Esseen bound for Student's statistic, Ann. Probab. 24 (1996) 491–503.
[3] P. Billingsley, Convergence of Probability Measures, Wiley, New York, 1968.
[4] G.A. Brosamler, An almost everywhere central limit theorem, Math. Proc. Cambridge Philos. Soc. 104 (1988) 561–574.
[5] Y.S. Chow, H. Teicher, Probability Theory: Independence, Interchangeability, Martingales, second ed., Springer, New York, 1988.
[6] M. Csörgő, B. Szyszkowicz, Q.Y. Wang, Darling–Erdös theorem for self-normalized partial sums, Ann. Probab. 31 (2003) 676–692.
[7] M. Csörgő, B. Szyszkowicz, Q.Y. Wang, Donsker's theorem for self-normalized partial sums processes, Ann. Probab. 31 (2003) 1228–1240.

[8] E. Giné, F. Götze, D.M. Mason, When is the Student t-statistic asymptotically standard normal?, Ann. Probab. 25 (1997) 1514–1531.
[9] K. Gonchigdanzan, An almost sure limit theorem for the product of partial sums with stable distribution, Statist. Probab. Lett. 78 (18) (2008) 3170–3175.
[10] K. Gonchigdanzan, Almost sure functional limit theorem for the product of partial sums, ESAIM Probab. Stat. (2008), doi:10.1051/ps:2008038.
[11] K. Gonchigdanzan, G. Rempala, A note on the almost sure limit theorem for the product of partial sums, Appl. Math. Lett. 19 (2006) 191–196.
[12] P.S. Griffin, J.D. Kuelbs, Self-normalized laws of the iterated logarithm, Ann. Probab. 17 (1989) 1571–1601.
[13] Y.X. Li, J.F. Wang, An almost sure central limit theorem for products of sums under association, Statist. Probab. Lett. 78 (4) (2008) 367–375.
[14] Z.Y. Lin, A self-normalized Chung-type law of iterated logarithm, Theory Probab. Appl. 41 (1996) 791–798.
[15] W.D. Liu, Z.Y. Lin, Asymptotics for self-normalized random products of sums for mixing sequences, Stoch. Anal. Appl. 25 (2007) 293–315.
[16] X. Lu, Y. Qi, A note on asymptotic distribution for products of sums, Statist. Probab. Lett. 68 (2004) 407–413.
[17] T.X. Pang, Z.Y. Lin, K.S. Hwang, Asymptotics for self-normalized random products of sums of i.i.d. random variables, J. Math. Anal. Appl. 334 (2007) 1246–1259.
[18] M. Peligrad, Q.M. Shao, A note on the almost sure central limit theorem, Statist. Probab. Lett. 22 (1995) 131–136.
[19] Y. Qi, Limit distributions for products of sums, Statist. Probab. Lett. 62 (2003) 93–100.
[20] G. Rempala, J. Wesolowski, Asymptotic for products of sums and U-statistics, Electron. Comm. Probab. 7 (2002) 47–54.
[21] P. Schatte, On strong versions of the central limit theorem, Math. Nachr. 137 (1988) 249–256.
[22] Q.M. Shao, Self-normalized large deviations, Ann. Probab. 25 (1997) 285–328.
[23] Q.M. Shao, Self-normalized large deviations, in: Asymptotic Methods in Probability and Statistics, Elsevier Science, 1998, pp. 467–480.
[24] Q.M. Shao, A Cramér type large deviation result for Student's t-statistics, J. Theoret. Probab. 12 (1999) 385–398.
[25] Y. Zhang, X.Y. Yang, Z.S. Dong, An almost sure central limit theorem for products sums of partial sums under association, J. Math. Anal. Appl. 355 (2) (2009) 708–716.